

Formulation, existence, and computation of simultaneous route-and-departure choice bounded rationality dynamic user equilibrium with fixed or endogenous user tolerance

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Abstract

This paper analyzes the continuous-time *simultaneous route-and-departure choice* (SRDC) *dynamic user equilibrium* (DUE) that incorporates boundedness of user rationality. As such, the *bounded rationality* dynamic user equilibrium (BR-DUE) model assumes that travelers do not always seek the least costly route-and-departure-time choice. Rather, their perception of travel cost is affected by an indifference band which describes travelers' tolerance towards the difference between their experienced travel costs and the minimum travel cost. We further extend our subject of investigation to include a *variable tolerance* (VT) BR-DUE with endogenously determined tolerances that may depend not only on a particular path, but also on the established path flows (path departure rates). The VT-BR-DUE model captures more realistic driving behaviors that stem from path heterogeneity, and drivers' observations of prevailing traffic conditions.

For the first time in the literature, the continuous-time VT-BR-DUE problem, together with the BR-DUE problem as a special case, is formulated as an infinite dimensional variational inequality, a differential variational inequality, and a fixed point problem. Existence results for VT-BR-DUE and BR-DUE are provided based on assumptions weaker than those made for SRDC DUEs. Moreover, we propose, based on the fixed point formulation, a fixed-point iterative algorithm that computes VT-BR-DUE and BR-DUE in continuous time, which is then tested on several networks in terms of solution characteristics, convergence, and computational time.

Keywords: simultaneous route-and-departure choice dynamic user equilibrium; bounded rationality; variable tolerance; variational inequality; differential variational inequality; existence; computation

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1 Introductory remarks

This paper studies an extension of the *simultaneous route-and-departure choice dynamic user equilibrium* (SRDC DUE) articulated first by Friesz et al. (1993) and discussed subsequently by Friesz and Mookherjee (2006); Friesz et al. (2011, 2013); and Friesz and Meimand (2013). Namely we incorporate the concept of *bounded rationality* (BR) proposed by Simon (1957, 1990, 1991) for the modeling of travel behavior. As such, BR-DUE models are developed under the assumption that travelers, viewed as Nash agents, do not behave in a completely rational manner.

In the literature of traffic user equilibrium, the modeling of travelers’ route and/or departure time choices has been greatly influenced by Wardrop’s first principle (Wardrop, 1952), which states that road users behave in a completely rational way and seek to minimize their own travel time by making route (or departure time) choices. There are multiple means of expressing the dynamic notion of Wardropian user equilibrium, such as variational inequality (Friesz et al., 1993; Han et al., 2013c; Szeto and Lo, 2004), differential variational inequality (Friesz et al., 2001, 2011, 2013), and nonlinear complementarity problem (Han et al., 2011; Wie et al., 2002). While admitting a number of canonical mathematical representations, the notion of complete rationality user equilibrium may enjoy less support from the viewpoint of realistic driving behavior and from empirical evidence. That is, travelers may not always choose the departure time and route that yield the minimum travel cost. Such a situation could be due to (1) imperfect travel information; and (2) certain “inertia” in decision-making. Moreover, empirical studies suggest that in reality, drivers do not always follow the least costly route-and-departure-time choice (Avineri and Prashker, 2004).

1.1 Some literature on bounded rationality

As a relaxation of the otherwise restrictive Wardropian assumption, the notion of bounded rationality is proposed by Simon (1957, 1990, 1991) and introduced to traffic modeling by Mahmassani and Chang (1987). In prose, the notion of bounded rationality postulates a range of acceptable travel costs that, when achieved, do not incentivize travelers to change their departure times or route choices. Such a range is phrased by Mahmassani and Chang (1987) as “indifference band”. The width of such band, usually denoted by ε , is either derived through a behavioral study of road users (for example, by surveys) or calibrated from empirical observation through inverse modeling techniques. In general, ε could depend on a specific origin-destination pair and/or travel commodity. Bounded rationality user equilibrium has gradually become a major field of inquiry especially in *static traffic assignment* (STA), with an incomplete list of papers including Di et al. (2013); Gifford and Checherita (2007); Han and Timmermans (2006); Khisty and Arslan (2005); Luo et al. (2010) and Marsden et al. (2012). It is also investigated via simulation-based approaches in the venue of dynamic modeling (Hu and Mahmassani, 1997; Mahmassani and Jayakrishnan, 1991; Mahmassani and Liu, 1999; Mahmassani et al., 2005).

Although the concept of bounded rationality has been used extensively in STA and simulation models, there have been very few analytical results regarding the dynamic notion of bounded rationality user equilibria. Szeto (2003) and Szeto and Lo (2006) are the first to propose bounded rationality dynamic user equilibrium (BR-DUE), although that study does not accommodate drivers’ departure time choices. In Szeto and Lo (2006), a *route-choice* (RC) BR-DUE is formulated as a discrete-time nonlinear complementarity problem. They propose a heuristic route-swapping algorithm adapted from Huang and Lam (2002) to solve the RC

BR-DUE. Ge and Zhou (2012) consider RC BR-DUE with endogenously determined tolerances by allowing the width of the indifference band ε to depend on time and the actual path flows. However, no solution method is proposed in that paper. To the best of our knowledge, there has not been any analytical treatment of the simultaneous route-and-departure choice BR-DUE with exogenous tolerances or with variable (endogenous) tolerance (VT-BR-DUE) in the literature, in terms of formulation, qualitative properties, and computation. This paper successfully bridges such gap in the literature.

1.2 Contribution made in this paper

Study of the BR-DUE problem is most facilitated by formulating it into a canonical mathematical form so that existing analytical and computational tools may apply. This paper proposes, for the first time in the literature, three equivalent mathematical formulations of the SRDC BR-DUE problem with exogenously given tolerances, namely a variational inequality (VI), a differential variational inequality (DVI), and a fixed point problem (FPP). Remarkably, we are able to establish the same analytical formulations for a generalization of the SRDC BR-DUE problem, in which the tolerance ε could depend not only on the O-D pair but also on the path and the underlying path flows. Such generalized version, which we call the variable tolerance (VT) BR-DUE, endogenizes the user tolerances and subsumes the SRDC BR-DUE as a special case¹. The rationale behind the VT-BR-DUE model is related to the observation that in reality, drivers' tolerances towards the differences in travel costs may be affected by not only user class distinguished by the O-D pair, but also by a number of other factors. For example, these tolerances may vary according to the path selected, since drivers' route decisions depend, in addition to their experienced travel times and arrival penalties, some other factors like road quality, information accessibility, and geographical characteristics etc., all of which are associated with a given path. Moreover, drivers' perception of travel cost may be influenced by the observation of the prevailing traffic state, which in turn is determined by the established path flows (path departure rates). For example, drivers taking the so-called "hot route" (Li et al., 2007) (a geographically favorable path with significant traffic volume) is more likely to stick with the current route than those taking some minor routes unless there is a significant difference in the experienced travel costs. Moreover, information availability and reliability in travelers' decision-making process are closely related to the traffic volume on each path (Herrera et al., 2009). In any of the scenarios depicted above, the tolerance depends on the path and/or the established path flows. The variable tolerance BR-DUE considered in this paper is meant to capture these more realistic behavioral aspects of network modeling. Notice that this paper treats the dependence of the cost tolerance on the path and path flows in the most general way possible; the established results are readily applicable to any specific form of such dependence, which can be obtained from further study on model calibration by means of surveys, inverse modeling, and so forth.

The concept of variable tolerance is first proposed by Ge and Zhou (2012) in their study of *route choice* (RC) BR-DUE, but that paper does not address the notion of simultaneous route-and-departure choice, nor does it provide discussions on the existence or computation of VT-BR-DUE, whereas this paper addresses all of these issues by means of a measure-theoretic argument and the optimal control theory. The most significant contribution made by this paper is the formulation of the VT-BR-DUE problem, together with the BR-DUE problem as a special case, as equivalent mathematically canonical forms, namely, as VI, DVI

¹Due to the fact that BR-DUE is a special case of VT-BR-DUE, we will, in the rest of this paper, state results only for VT-BR-DUE, noting that the same results apply to BR-DUE automatically.

and FPP. As we explain, the proposed infinite-dimensional variational inequality formulation accommodates the users' tolerance, which may depend on exogenous network parameters and endogenous system states, by incorporating such tolerance into the *effective path delay operator* (Friesz et al., 1993; Han, 2013). It is significant that the proposed analytic framework can capture all the aforementioned realistic behavioral features of an urban traffic network, and remains mathematically internally consistent without *ad hoc* treatment of travel delays, departure time choices, flow propagation or other critical model features.

The most obvious approach to establishing existence for VT-BR-DUE is to convert the problems to an equivalent variational inequality problem and then apply Brouwer's fixed point existence theorem (Browder, 1968). Approaches based on Brouwer's theorem require the set of feasible path flows (departure rates) under consideration to be compact and convex in a topological vector space (Han et al., 2013c), and typically involve an *a priori* bound on all the path flows (Zhu and Marcotte, 2000).

Notice that the existence of an SRDC DUE trivially implies the existence of a VT-BR-DUE or a BR-DUE, thus a meaningful discussion of existence for bounded rationality DUEs should be based on conditions weaker than those for SRDC DUEs (such as those provided in Han et al. (2013c)). This is achieved in this paper. In particular, we will first review the existence result for SRDC DUE established by Han et al. (2013c), which is arguably the most general existence result in the literature as it is based on very minor conditions on the travel cost functions and involves no *a priori* boundedness of path flows. In fact, as we will show later, a violation of the conditions stipulated by Han et al. (2013c) will indeed result in the non-existence of SRDC DUEs. We will then provide existence results for VT-BR-DUE with conditions weaker than those assumed by Han et al. (2013c) for the SRDC DUE case, and therefore claiming a more general existence result for VT-BR-DUE.

On the computational side, prior to this paper there has not been any analytical methods proposed in the literature for computing VT-BR-DUE or BR-DUE, when both route and departure time are within the purview of drivers. The analytical framework put forward by this paper has a significant impact on the computation of SRDC VT-BR-DUEs, given many existing computational algorithms for variational inequalities (VIs), differential variational inequalities (DVI) and fixed point problems (FPP) in the literature. The proposed VI formulation for BR-DUE, when properly discretized, admits a number of existing numerical algorithms (Huang and Lam, 2002; Szeto and Lo, 2004; Ukkusuri et al., 2012). On the other hand, the computation of a continuous-time VT-BR-DUE is most facilitated by the mathematical paradigm of DVI (Pang and Stewart, 2008) and emerging computational algorithms associated therein. In particular, we propose in this paper a continuous-time fixed point algorithm for VT-BR-DUE, which is derived from the necessary conditions of a fictitious optimal control problem equivalent to the DVI formulation of VT-BR-DUE.

In summary, contributions made by this paper include:

- expression of the simultaneous route-and-departure choice (SRDC) dynamic user equilibrium with bounded rationality (BR-DUE) and with variable tolerance (VT-BR-DUE) as variational inequalities, differential variational inequalities, and fixed point problems in Hilbert spaces;
- existence result for the SRDC VT-BR-DUE under conditions weaker than those ensuring existence of SRDC DUE (Han et al., 2013c); and
- a fixed-point algorithm for computing SRDC VT-BR-DUE in continuous time, which is

tested on several networks in terms of solution characteristics, convergence, and computational time.

It should be mentioned that although we are mainly interested in SRDC DUE problems incorporating bounded user rationality, the proposed analytical framework is easily transferrable to route-choice BR-DUE and to static BR-UE (bounded rationality user equilibrium) problems. This will be demonstrated in some other work.

1.3 Organization

The rest of this paper is organized as follows. Section 2 introduces basic notations and background materials necessary for the presentation of subsequent analyses. It also articulates the definitions of SRDC DUE, BR-DUE and VT-BR-DUE. The proposed VI formulation for the SRDC VT-BR-DUE will be established in Section 3. Section 4 presents the DVI and FPP formulations of the SRDC VT-BR-DUE problem. A fixed-point algorithm will also be derived from the FPP formulation. In Section 5 we provide an in-depth discussion on the existence of SRDC VT-BR-DUE. Section 6 presents several computational results based on the proposed fixed-point algorithm. Finally, Section 7 offers some concluding remarks.

2 Notation, essential background, and definition

Throughout this paper, the time interval of analysis is a single commuting period or “day” expressed as

$$[t_0, t_f] \subset \mathfrak{R}$$

where $t_f > t_0$, and both t_0 and t_f are fixed. Here, as in all DUE modeling, the single most crucial ingredient is the path delay operator, which provides the delay on any path p per unit of flow departing from the origin of that path; it is denoted by

$$D_p(t, h) \quad \forall p \in \mathcal{P}$$

where \mathcal{P} is the set of all paths employed by travelers, t denotes departure time, and h is a vector of departure rates. From these, we construct effective unit path delay operators $\Psi_p(t, h)$ by adding the so-called schedule delay $f(t + D_p(t, h) - T_A)$; that is

$$\Psi_p(t, h) = D_p(t, h) + f(t + D_p(t, h) - T_A) \quad \forall p \in \mathcal{P} \quad (2.1)$$

where T_A is the desired arrival time and $T_A < t_f$. The function $f(\cdot)$ assesses a penalty whenever

$$t + D_p(t, h) \neq T_A \quad (2.2)$$

since $t + D_p(t, h)$ is the clock time at which departing traffic arrives at the destination of path $p \in \mathcal{P}$. Note that, for convenience, T_A is assumed to be independent of destination. However, that assumption is easy to relax, and the consequent generalization of our model is a trivial extension. We interpret $\Psi_p(t, h)$ as the perceived travel cost of drivers starting at time t on path p under travel conditions h . We stipulate that each

$$\Psi_p(\cdot, h) : [t_0, t_f] \longrightarrow \mathfrak{R}_{++} \quad \forall p \in \mathcal{P}$$

is measurable, strictly positive, and square integrable. We employ the obvious notation

$$\Psi(\cdot, h) \doteq (\Psi_p(\cdot, h) : p \in \mathcal{P}) \in \mathfrak{R}_{++}^{|\mathcal{P}|}$$

to express the complete vector of effective delay operators.

It is now well known that path delay operators may be obtained from an embedded delay model, data combined with response surface methodology, or data combined with inverse modeling. Unfortunately, regardless of how derived, realistic path delay operators do not possess the desirable property of monotonicity; they may also be non-differentiable.

2.1 Simultaneous route-and-departure choice dynamic user equilibrium (SRDC DUE)

We recap in this subsection the notion of SRDC DUE originally articulated by Friesz et al. (1993).

We introduce the fixed trip matrix $(Q_{ij} : (i, j) \in \mathcal{W})$, where each $Q_{ij} \in \mathbb{R}_+$ is the fixed travel demand between origin-destination pair $(i, j) \in \mathcal{W}$, where \mathcal{W} is the set of origin-destination pairs. Note that Q_{ij} represents traffic volume, not flow. Finally we let $\mathcal{P}_{ij} \subset \mathcal{P}$ be the set of paths connecting origin-destination pair $(i, j) \in \mathcal{W}$. As mentioned earlier h is the vector of path flows $h = (h_p : p \in \mathcal{P})$. We denote the space of square integrable functions on the real interval $[t_0, t_f]$ by $L^2[t_0, t_f]$. We stipulate that each path flow is square integrable, that is

$$h \in (L_+^2[t_0, t_f])^{|\mathcal{P}|}$$

where $(L_+^2[t_0, t_f])^{|\mathcal{P}|}$ is the non-negative cone of the $|\mathcal{P}|$ -fold product of the Hilbert spaces $L^2[t_0, t_f]$. The inner product on the Hilbert space $(L^2[t_0, t_f])^{|\mathcal{P}|}$ is defined as

$$\langle u, v \rangle = \int_{t_0}^{t_f} (u(s))^T v(s) ds = \sum_{i=1}^{|\mathcal{P}|} \int_{t_0}^{t_f} u_i(s) v_i(s) ds \quad (2.3)$$

where the superscript T denotes transpose of vectors. Moreover, the norm

$$\|u\|_{L^2} = \langle u, u \rangle^{1/2} \quad (2.4)$$

is induced by the inner product (2.3).

Each element in $(L_+^2[t_0, t_f])^{|\mathcal{P}|}$, which is of the form $h = (h_p : p \in \mathcal{P})$, is interpreted as a vector of departure rates, or more simply path flows, measured at the entrance of the first arc of the relevant path. It will be seen that these departure rates are defined only up to a set of measure zero. With this in mind, let ν denote a Lebesgue measure on $[t_0, t_f]$, and let $\forall_\nu(t) \in [t_0, t_f]$ denote the phrase *for ν -almost every $t \in [t_0, t_f]$* .

We write the flow conservation constraints as

$$\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t) dt = Q_{ij} \quad \forall (i, j) \in \mathcal{W} \quad (2.5)$$

where (2.5) consists of Lebesgue integrals. Using the notation and concepts we have mentioned, the feasible region for the DUE problem is

$$\Lambda_0 = \left\{ h \geq 0 : \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t) dt = Q_{ij} \quad \forall (i, j) \in \mathcal{W} \right\} \subseteq (L_+^2[t_0, t_f])^{|\mathcal{P}|} \quad (2.6)$$

In order to define an appropriate concept of minimum travel costs in the present context, we require the measure-theoretic analog of the infimum of a set of numbers. In particular, for any measurable function $g : [t_0, t_f] \rightarrow \mathbb{R}$, the *essential infimum* of $g(\cdot)$ on $[t_0, t_f]$ is given by

$$\text{essinf} \{g(s) : s \in [t_0, t_f]\} \doteq \sup \{x \in \mathbb{R} : \nu\{s \in [t_0, t_f] : g(s) < x\} = 0\} \quad (2.7)$$

Note that for each $x > \text{essinf}\{g(s) : s \in [t_0, t_f]\}$ it must be true by definition that

$$\nu\{s \in [t_0, t_f] : f(s) < x\} > 0$$

Let us define the essential infimum of effective travel delays, which depend on the path flows h :

$$v_p(h) = \text{essinf}\{\Psi_p(t, h) : t \in [t_0, t_f]\} > 0 \quad \forall p \in \mathcal{P} \quad (2.8)$$

$$v_{ij}(h) = \min\{v_p(h) : p \in \mathcal{P}_{ij}\} \quad \forall (i, j) \in \mathcal{W} \quad (2.9)$$

The following definition of dynamic user equilibrium is first articulated by Friesz et al. (1993):

Definition 2.1. (SRDC dynamic user equilibrium) *A vector of departure rates (path flows) $h^* \in \Lambda_0$ is a dynamic user equilibrium with simultaneous route-and-departure choice if*

$$h_p^*(t) > 0, \quad p \in \mathcal{P}_{ij} \implies \Psi_p(t, h^*) = v_{ij}(h^*) \quad \forall \nu(t) \in [t_0, t_f] \quad (2.10)$$

We denote this equilibrium by $DUE(\Psi, \Lambda_0, [t_0, t_f])$.

Using measure-theoretic arguments, Friesz et al. (1993) establish that a dynamic user equilibrium is equivalent to the following variational inequality under suitable regularity conditions:

$$\left. \begin{array}{l} \text{find } h^* \in \Lambda_0 \text{ such that} \\ \sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Psi_p(t, h^*)(h_p - h_p^*) dt \geq 0 \\ \forall h \in \Lambda_0 \end{array} \right\} VI(\Psi, \Lambda_0, [t_0, t_f]) \quad (2.11)$$

2.2 SRDC BR-DUE

The notion of bounded rationality (BR) is a relaxation of Wardrop's first principle (Wardrop, 1952). The Wardropian principle incorporating the simultaneous route-and-departure notion requires identical travel costs among all utilized routes and departure time choices between an origin-destination pair. The bounded rationality dynamic user equilibrium (BR-DUE), on the other hand, requires that the experience travel cost, including early and late arrival penalties, are within the interval $[v_{ij}(h), v_{ij}(h) + \varepsilon_{ij}]$, where $v_{ij}(h)$ denotes the essential infimum of the effective delays between origin-destination pair $(i, j) \in \mathcal{W}$, and \mathcal{W} is the set of O-D pairs in the traffic network. Notice that such infimum depends on the complete path flow vector h . $\varepsilon_{ij} \in \mathbb{R}_+$ is a prescribed constant specific to O-D pair (i, j) , which describes an acceptable difference in travel cost experienced by a traveler between O-D pair (i, j) .

Throughout this subsection, the travel demand Q_{ij} between each origin-destination pair (i, j) is assumed to be a fixed constant. Recalling the effective path delay operator (2.1) and the feasible path flow set (2.6), we articulate the notion of SRDC BR-DUE in a measure-theoretic framework as follows.

Definition 2.2. (SRDC BR-DUE) *Given $\varepsilon = (\varepsilon_{ij} : (i, j) \in \mathcal{W}) \in \mathbb{R}_+^{|\mathcal{W}|}$, a vector of departure rates $h^* \in \Lambda_0$ is a bounded rationality dynamic user equilibrium associated with the vector of tolerances ε if for all $(i, j) \in \mathcal{W}$,*

$$h_p^*(t) > 0, \quad p \in \mathcal{P}_{ij} \implies \Psi_p(t, h^*) \in [v_{ij}(h^*), v_{ij}(h^*) + \varepsilon_{ij}] \quad \forall \nu(t) \in [t_0, t_f] \quad (2.12)$$

where $v_{ij}(h^*)$ is the essential infimum of the effective path delays between origin-destination pair (i, j) . We denote this equilibrium by $BR-DUE(\Psi, \varepsilon, \Lambda_0, [t_0, t_f])$.

2.3 SRDC VT-BR-DUE

Generalization of the BR-DUE in Definition 2.2 to accommodate variable tolerance is straightforward by allowing the constant ε_{ij} , $(i, j) \in \mathcal{W}$ to depend on path and path departure rates. In particular, we introduce $\varepsilon_{ij}^p(h) \in \mathbb{R}_+$ for $(i, j) \in \mathcal{W}$, $p \in \mathcal{P}_{ij}$, and $h \in \Lambda_0$, where \mathcal{W} denotes the set of O-D pairs, \mathcal{P}_{ij} denotes the set of paths connecting origin i to destination j , and Λ_0 is defined in (2.6). Define $\varepsilon(h) \doteq (\varepsilon_{ij}^p(h) : (i, j) \in \mathcal{W}, p \in \mathcal{P}_{ij})$ to be the concatenation of tolerances associated with each path. Note that $\varepsilon(\cdot)$ is viewed as a mapping from Λ_0 , the set of feasible path flows, to $\mathbb{R}_+^{|\mathcal{P}|}$.

Definition 2.3. (SRDC VT-BR-DUE) *Given $\varepsilon(\cdot) : \Lambda_0 \rightarrow \mathbb{R}_+^{|\mathcal{P}|}$, a vector of departure rates $h^* \in \Lambda_0$ is a bounded rationality dynamic user equilibrium with variable tolerance $\varepsilon(t, h)$ if for all $(i, j) \in \mathcal{W}$,*

$$h_p^*(t) > 0, \quad p \in \mathcal{P}_{ij} \implies \Psi_p(t, h^*) \in [v_{ij}(h^*), v_{ij}(h^*) + \varepsilon_{ij}^p(h^*)] \quad \forall \nu(t) \in [t_0, t_f] \quad (2.13)$$

where $v_{ij}(h^*)$ is the essential infimum of the effective path delays between origin-destination pair (i, j) . We denote this equilibrium by VT-BR-DUE($\Psi, \varepsilon, \Lambda_0, [t_0, t_f]$).

Remark 2.4. *It can be easily seen that the BR-DUE is just one special case of the VT-BR-DUE problem, in which the dependence of $\varepsilon_{ij}^p(h)$ on p and h are dropped. Thus, to simultaneously analyze both models using the proposed methodological framework, it suffices for us to treat the VT-BR-DUE only, and the established results will automatically hold for the BR-DUE problem.*

3 The VI formulation for the SRDC VT-BR-DUE and BR-DUE problems

In this section, we present the infinite-dimensional variational inequality (VI) formulation for the BR-DUE and VT-BR-DUE problems, when both route choice and departure time choice are employed. These formulations are completely original, and will lead to the qualitative analyses presented later on the existence and computation of these two models. For the reason stated in Remark 2.4, we will state and prove the VI formulation for the VT-BR-DUE problem, followed by a lemma that expresses the VI formulation for the BR-DUE problem.

In the formulation of the VT-BR-DUE problem, the vector of tolerances $(\varepsilon_{ij}^p(h) : (i, j) \in \mathcal{W}, p \in \mathcal{P}_{ij}) \in \mathbb{R}_+^{|\mathcal{P}|}$ is assumed to depend on the vector of path flows h . Essential to the variational inequality formulation of the VT-BR-DUE problem is the following operator:

$$\Phi^\varepsilon : \Lambda_0 \rightarrow (L_+^2[t_0, t_f])^{|\mathcal{P}|}, \quad h \mapsto (\Phi_p^\varepsilon(\cdot, h) : p \in \mathcal{P}) \quad (3.14)$$

where

$$\Phi_p^\varepsilon(t, h) = \max \left\{ \Psi_p(t, h), v_{ij}(h) + \varepsilon_{ij}^p(h) \right\} - \left(\varepsilon_{ij}^p(h) - \min_{q \in \mathcal{P}_{ij}} \left\{ \varepsilon_{ij}^q(h) \right\} \right) \quad \forall p \in \mathcal{P}_{ij} \quad (3.15)$$

Given any vector of path flows $h \in \Lambda_0$, the effective path delays $\Psi_p(t, h)$, $p \in \mathcal{P}$, the minimum effective delays $v_{ij}(h)$, $(i, j) \in \mathcal{W}$, and the tolerances $\varepsilon_{ij}^p(h)$, $(i, j) \in \mathcal{W}$, $p \in \mathcal{P}_{ij}$ are all uniquely determined by the dynamic network loading (DNL) procedure without any ambiguity. Thus, the operator Φ^ε stated above is indeed a well-defined mapping from Λ_0 into

$(L_+^2[t_0, t_f])^{|\mathcal{P}|}$. We indicate the dependence of such an operator on the prescribed function $\varepsilon(\cdot)$ by a superscript.

The next theorem casts the $VT\text{-}BR\text{-}DUE(\Psi, \varepsilon, \Lambda_0, [t_0, t_f])$ problem as an infinite-dimensional variational inequality problem.

Theorem 3.1. (VT-BR-DUE equivalent to a variational inequality) *Given $\varepsilon(\cdot) : \Lambda_0 \rightarrow \mathbb{R}_+^{|\mathcal{P}|}$, let $\Phi_p^\varepsilon(\cdot, h) : [t_0, t_f] \rightarrow \mathbb{R}_{++}$ be measurable and strictly positive for all $p \in \mathcal{P}$ and all $h \in \Lambda_0$, given by (3.14) and (3.15). Then a vector $h^* \in \Lambda_0$ represents a $VT\text{-}BR\text{-}DUE(\Psi, \varepsilon, \Lambda_0, [t_0, t_f])$ if and only if it solves the following variational inequality*

$$\left. \begin{aligned} & \text{find } h^* \in \Lambda_0 \text{ such that} \\ & \sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Phi_p^\varepsilon(t, h^*)(h_p - h_p^*) dt \geq 0 \\ & \forall h \in \Lambda_0 \end{aligned} \right\} VI(\Phi^\varepsilon, \Lambda_0, [t_0, t_f]) \quad (3.16)$$

Proof. (i) [Necessity] Let $h^* \in \Lambda_0$ correspond to a $VT\text{-}BR\text{-}DUE$ solution, then for any $h \in \Lambda_0$ and any $(i, j) \in \mathcal{W}$,

$$\begin{aligned} & \sum_{(i,j) \in \mathcal{W}} \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \Phi_p^\varepsilon(t, h^*) h_p(t) dt \geq \sum_{(i,j) \in \mathcal{W}} \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \mu_{ij}^\varepsilon(h^*) h_p(t) dt \\ & = \sum_{(i,j) \in \mathcal{W}} \mu_{ij}^\varepsilon(h^*) \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t) dt = \sum_{(i,j) \in \mathcal{W}} \mu_{ij}^\varepsilon(h^*) Q_{ij} \end{aligned} \quad (3.17)$$

where

$$\mu_{ij}^\varepsilon(h^*) \doteq \min \{ \mu_p^\varepsilon(h^*) : p \in \mathcal{P}_{ij} \} \quad (3.18)$$

and

$$\mu_p^\varepsilon(h^*) \doteq \text{essinf} \{ \Phi_p^\varepsilon(t, h^*) : t \in [t_0, t_f] \} \quad (3.19)$$

It is easy to deduce that $\mu_{ij}^\varepsilon(h^*) = v_{ij}(h^*) + \min_{p \in \mathcal{P}_{ij}} \{ \varepsilon_{ij}^p(h^*) \}$. On the other hand, in view of (2.13) and (3.15), we perform the following deduction for every $p \in \mathcal{P}_{ij}$:

$$\begin{aligned} h_p^*(t) > 0 & \implies \Psi_p(t, h^*) \leq v_{ij}(h^*) + \varepsilon_{ij}^p(h^*) \\ & \implies \Phi_p^\varepsilon(t, h^*) = v_{ij}(h^*) + \varepsilon_{ij}^p(h) - \left(\varepsilon_{ij}^p(h^*) - \min_{q \in \mathcal{P}_{ij}} \{ \varepsilon_{ij}^q(h^*) \} \right) = \mu_{ij}^\varepsilon(h^*) \end{aligned}$$

Therefore, according to the non-negativity of h , we have

$$\sum_{(i,j) \in \mathcal{W}} \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \Phi_p^\varepsilon(t, h^*) h_p^*(t) dt = \sum_{(i,j) \in \mathcal{W}} \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \mu_{ij}^\varepsilon(h^*) h_p^*(t) dt = \sum_{(i,j) \in \mathcal{W}} \mu_{ij}^\varepsilon(h^*) Q_{ij} \quad (3.20)$$

In view of (3.17) and (3.20), we have

$$\sum_{(i,j) \in \mathcal{W}} \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \Phi_p^\varepsilon(t, h^*) h_p^*(t) dt \leq \sum_{(i,j) \in \mathcal{W}} \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \Phi_p^\varepsilon(t, h^*) h_p(t) dt \quad \forall h \in \Lambda_0$$

which is recognized as the variational inequality (3.16).

(ii) [Sufficiency] Let $h^* \in \Lambda_0$ be a solution of the variational inequality $VI(\Phi^\varepsilon, \Lambda_0, [t_0, t_f])$. Given that $\Phi^\varepsilon : \Lambda_0 \rightarrow (L_+^2[t_0, t_f])^{|\mathcal{P}|}$ is measurable and strictly positive, we employ the same proof of Theorem 2 from Friesz et al. (1993) to show that h^* satisfies

$$h_p^*(t) > 0, p \in \mathcal{P}_{ij} \implies \Phi_p^\varepsilon(t, h^*) = \mu_{ij}^\varepsilon(h^*) \quad \forall_\nu(t) \in [t_0, t_f], \quad \forall(i, j) \in \mathcal{W} \quad (3.21)$$

where $\mu_{ij}^\varepsilon(h^*)$ is given by (3.18) and is equal to $v_{ij}(h^*) + \min_{p \in \mathcal{P}_{ij}} \{\varepsilon_{ij}^p(h^*)\}$. We readily deduce that

$$\begin{aligned} h_p^*(t) > 0, p \in \mathcal{P}_{ij} &\implies \Phi_p^\varepsilon(t, h^*) = v_{ij}(h^*) + \min_{p \in \mathcal{P}_{ij}} \{\varepsilon_{ij}^p(h^*)\} \\ \implies \max \left\{ \Psi_p(t, h^*), v_{ij}(h^*) + \varepsilon_{ij}^p(h^*) \right\} - \varepsilon_{ij}^p(h^*) + \min_{q \in \mathcal{P}_{ij}} \{\varepsilon_{ij}^q(h^*)\} &= v_{ij}(h^*) + \min_{q \in \mathcal{P}_{ij}} \{\varepsilon_{ij}^q(h^*)\} \\ \implies \max \left\{ \Psi_p(t, h^*), v_{ij}(h^*) + \varepsilon_{ij}^p(h^*) \right\} &= v_{ij}(h^*) + \varepsilon_{ij}^p(h^*) \\ \implies v_{ij}(h^*) \leq \Psi_p^\varepsilon(t, h^*) \leq v_{ij}(h^*) + \varepsilon_{ij}^p(h^*) &\quad \forall_\nu(t) \in [t_0, t_f], \quad \forall p \in \mathcal{P}_{ij}, \quad \forall(i, j) \in \mathcal{W} \end{aligned}$$

and conclude that h^* solves the VT-BR-DUE problem. \square

As a special case of Theorem 3.1, we present the VI formulation for the BR-DUE problem with exogenously given tolerances $\varepsilon = (\varepsilon_{ij} : (i, j) \in \mathcal{W}) \in \mathfrak{R}_+^{|\mathcal{W}|}$.

Lemma 3.2. (BR-DUE equivalent to a variational inequality) *Given the fixed tolerance vector $\varepsilon = (\varepsilon_{ij} : (i, j) \in \mathcal{W}) \in \mathfrak{R}_+^{|\mathcal{W}|}$, define*

$$\phi_p^\varepsilon(t, h) \doteq \max \{ \Psi_p(t, h), v_{ij}(h) + \varepsilon_{ij} \} \quad \forall p \in \mathcal{P}_{ij} \quad (3.22)$$

Then a vector $h^ \in \Lambda_0$ represents a BR-DUE($\Psi, \varepsilon, \Lambda_0, [t_0, t_f]$) if and only if it solves the following variational inequality*

$$\left. \begin{aligned} &\text{find } h^* \in \Lambda_0 \text{ such that} \\ &\sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \phi_p^\varepsilon(t, h^*) (h_p - h_p^*) dt \geq 0 \\ &\forall h \in \Lambda_0 \end{aligned} \right\} VI(\phi^\varepsilon, \Lambda_0, [t_0, t_f]) \quad (3.23)$$

Proof. In the case of BR-DUE with fixed tolerances ε_{ij} for each O-D pair $(i, j) \in \mathcal{W}$, the expression of the operator Φ_p^ε defined in (3.15) reduces to (3.22). The rest of the proof is the same as in Theorem 3.1. \square

In order to better understand the proposed variational inequality formulation for the BR-DUE problem, let us illustrate, in a less mathematically rigorous way, the meanings of $VI(\Psi, \Lambda_0, [t_0, t_f])$ (DUE) and $VI(\phi^\varepsilon, \Lambda_0, [t_0, t_f])$ (BR-DUE) using Figure 1 and Figure 2 respectively. For simplicity, let us consider just one O-D pair with only one path. Figure 1 depicts conceptually profiles of a DUE path flow $h_p^*(\cdot)$ and the corresponding effective path delay $\Psi_p(\cdot, h^*)$, both as functions of departure time t . It is intuitively clear that in order to minimize the following quantity over all $h_p \in \Lambda_0$,

$$\int_{t_0}^{t_f} \Psi_p(t, h^*) h_p(t) dt, \quad (3.24)$$

the non-zero portion (support) of $h_p^*(t)$ should be located at the “flat bottom” of the effective delay curve. Let us now turn to BR-DUE, which requires that whenever $h_p^*(t)$ is non-zero,

$\Psi_p(t, h^*)$ must be within an indifference band with width ε_{ij} . Clearly the upper boundary of such band is $v_{ij}(h^*) + \varepsilon_{ij}$. Thus, being a BR-DUE path flow requires that the non-zero portion must reside within the time interval $[a, b]$ (see Figure 2); i.e. the BR-DUE path flow minimizes

$$\int_{t_0}^{t_f} \phi_p^\varepsilon(t, h^*) h_p(t) dt \quad (3.25)$$

The case with the variable tolerances can be interpreted in a similar way, but is more difficult to visualize. Therefore we will omit it in this paper.

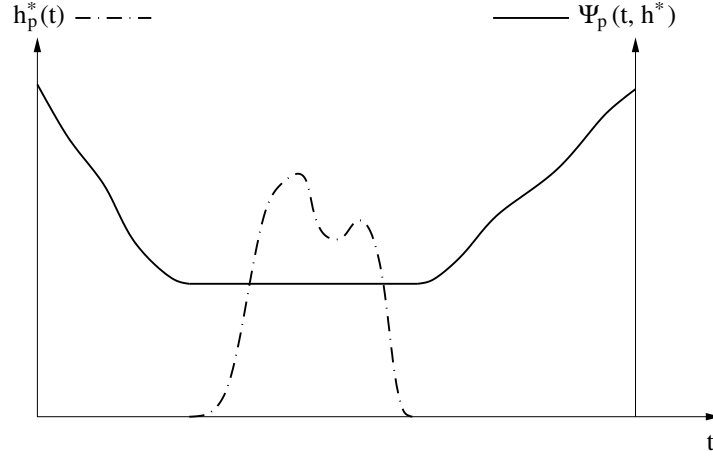


Figure 1: A possible scenario of a dynamic user equilibrium path flow $h_p^*(\cdot)$ and the associated effective path delay $\Psi_p(\cdot, h^*)$. In order to minimize the quantity (3.24) over all $h_p \in \Lambda_0$, the equilibrium path flow h_p^* must be located at the “flat bottom” of the effective delay curve. Such observation also coincides with the definition of DUE.

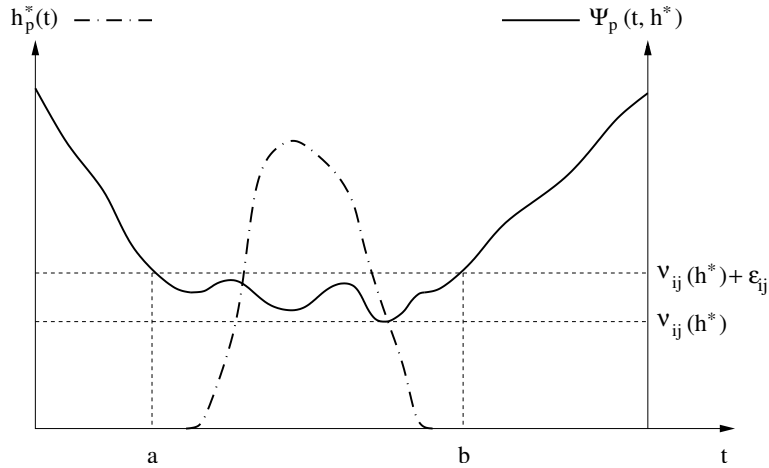


Figure 2: A possible scenario of a BR-DUE path flow $h_p^*(\cdot)$ and the associated effective path delay $\Psi_p(\cdot, h^*)$. To be a BR-DUE, h_p^* must vanish outside the time interval $[a, b]$, which implies that h_p^* is located at the “flat bottom” of the curve formed by the graph of $\Psi_p(\cdot, h^*)$ and the horizontal line $v_{ij}(h^*) + \varepsilon_{ij}$. Such a curve is precisely the graph of $\phi^\varepsilon(\cdot, h^*)$. Therefore, the BR-DUE is an analogy of the DUE when the operator Ψ is replaced with ϕ^ε .

Theorem 3.1 presents a variational inequality in its most canonical form, which is shown to be equivalent to the most general VT-BR-DUE problem. In comparison with the VI

formulation $VI(\Psi, \Lambda_0, [t_0, t_f])$ of SRDC DUE articulated by Friesz et al. (1993), the proposed variational inequality $VI(\Phi^\varepsilon, \Lambda_0, [t_0, t_f])$ relies on a modified principal operator Φ^ε , which encapsulates the dynamic network loading procedure as well as the variable tolerance mapping $\varepsilon(h)$. Notably, these two VIs do share the same mathematical structure which allows known methodologies regarding variational inequalities (Huang and Lam, 2002; Szeto and Lo, 2004; Ukkusuri et al., 2012) and SRDC DUE (Friesz et al., 1993, 2001, 2013, 2011; Friesz and Mookherjee, 2006; Han et al., 2013c; Han, 2013) to be transferred to the VT-BR-DUE and BR-DUE problems. In fact, a differential variational inequality (DVI) formulation and a fixed point problem (FPP) formulation of the VT-BR-DUE problem will become available, which will be presented in the following sections.

4 The DVI and FPP formulations of the SRDC VT-BR-DUE and BR-DUE problems

Let us introduce the function $y_{ij}(\cdot) : [t_0, t_f] \rightarrow [0, Q_{ij}]$ for each origin-destination pair (i, j) where Q_{ij} denotes the fixed travel demand between (i, j) . The quantity $y_{ij}(t)$ represents the total traffic volume that have departed from origin i with the intent of reaching destination j at time $t \in [t_0, t_f]$. Note that the set of feasible path flows Λ_0 defined in (2.6) can be re-stated as a two-point boundary value problem by means of isoperimetric constraints.

$$\Lambda_1 = \left\{ h \geq 0 : \frac{dy_{ij}}{dt} = \sum_{p \in \mathcal{P}_{ij}} h_p(t), \quad y_{ij}(t_0) = 0, \quad y_{ij}(t_f) = Q_{ij} \quad \forall (i, j) \in \mathcal{W} \right\} \quad (4.26)$$

As a result, we have the following equivalence theorem established for VT-BR-DUE, and the equivalence result for BR-DUE will come afterwards as a simple corollary.

Theorem 4.1. (VT-BR-DUE equivalent to a differential variational inequality)

Given $\varepsilon(\cdot) : \Lambda_1 \rightarrow \mathbb{R}_+^{|\mathcal{P}|}$, assume $\Phi_p^\varepsilon(\cdot, h)$, $t \in [t_0, t_f]$ is measurable and strictly positive for all $p \in \mathcal{P}$ and all $h \in \Lambda_1$, where the operator Φ_p^ε is defined in (3.14) and (3.15). A vector of departure rates (path flows) $h^* \in \Lambda_1$ is a variable tolerance bounded rationality dynamic user equilibrium if and only if h^* solves the following differential variational inequality

$$\left. \begin{aligned} & \text{find } h^* \in \Lambda_1 \text{ such that} \\ & \sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Phi_p^\varepsilon(t, h^*)(h_p - h_p^*) dt \geq 0 \\ & \forall h \in \Lambda_1 \end{aligned} \right\} DVI(\Phi^\varepsilon, \Lambda_1, [t_0, t_f]) \quad (4.27)$$

Proof. (i) [Sufficiency] Given that $h^* \in \Lambda_1$ is a solution of the differential variational inequality $DVI(\Phi^\varepsilon, \Lambda_1, [t_0, t_f])$, we have the following obvious relation:

$$\sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Phi_p^\varepsilon(t, h^*) h_p^*(t) dt \leq \sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Phi_p^\varepsilon(t, h^*) h_p(t) dt$$

for all $h \in \Lambda_1$. In other words, h^* is the solution of the following optimal control problem:

$$\min J(h) = \sum_{(i, j) \in \mathcal{W}} \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \Phi_p^\varepsilon(t, h^*) h_p(t) dt + \sum_{(i, j) \in \mathcal{W}} \mu_{ij} [Q_{ij} - y_{ij}(t_f)] \quad (4.28)$$

subject to

$$\frac{d}{dt}y_{ij}(t) = \sum_{p \in \mathcal{P}_{ij}} h_p(t) \quad \forall (i, j) \in \mathcal{W} \quad (4.29)$$

$$y_{ij}(t_0) = 0 \quad \forall (i, j) \in \mathcal{W} \quad (4.30)$$

$$h \geq 0 \quad (4.31)$$

where each μ_{ij} is the dual variable for the terminal condition on $y_{ij}(\cdot)$, $(i, j) \in \mathcal{W}$. The Hamiltonian for problem (4.28)-(4.31) is

$$H = \sum_{(i,j) \in \mathcal{W}} \sum_{p \in \mathcal{P}_{ij}} \Phi_p^\varepsilon(t, h^*) h_p + \sum_{(i,j) \in \mathcal{W}} \lambda_{ij} \sum_{p \in \mathcal{P}_{ij}} h_p$$

where the adjoint dynamics are:

$$\frac{d}{dt}\lambda_{ij}(t) = -\frac{\partial H}{\partial y_{ij}} = 0 \quad \forall (i, j) \in \mathcal{W}, \quad t \in [t_0, t_f] \quad (4.32)$$

the transversality conditions read:

$$\lambda_{ij}(t_f) = \frac{\partial \sum_{(i,j) \in \mathcal{W}} \mu_{ij} [Q_{ij} - y_{ij}(t_f)]}{\partial y_{ij}(t_f)} = -\mu_{ij} \quad \forall (i, j) \in \mathcal{W}, \quad t \in [t_0, t_f] \quad (4.33)$$

Taken together, (4.32) and (4.33) imply that

$$\lambda_{ij}(t) \equiv -\mu_{ij} \quad \forall (i, j) \in \mathcal{W}, \quad \forall t \in [t_0, t_f] \quad (4.34)$$

According to the minimum principle, we have

$$h^* = \arg \min_h H \quad \text{such that} \quad -h \leq 0$$

for which the Kuhn-Tucker conditions are

$$\Phi_p^\varepsilon(t, h^*) + \lambda_{ij} = \rho_p \geq 0 \quad \forall p \in \mathcal{P}_{ij}, \quad \forall (i, j) \in \mathcal{W}, \quad \forall t \in [t_0, t_f] \quad (4.35)$$

$$\rho_p(t) h_p^*(t) = 0 \quad \forall p \in \mathcal{P}_{ij}, \quad \forall (i, j) \in \mathcal{W}, \quad \forall t \in [t_0, t_f] \quad (4.36)$$

$$\rho_p(t) \geq 0 \quad \forall p \in \mathcal{P}_{ij}, \quad \forall (i, j) \in \mathcal{W}, \quad \forall t \in [t_0, t_f] \quad (4.37)$$

where ρ_p are dual variables for the non-negativity constraints. From (4.35) and (4.36) we have that for all $(i, j) \in \mathcal{W}$, $p \in \mathcal{P}_{ij}$,

$$h_p^*(t) > 0, p \in \mathcal{P}_{ij} \implies \Phi_p^\varepsilon(t, h^*) = -\lambda_{ij} = \mu_{ij} \quad \forall t \in [t_0, t_f] \quad (4.38)$$

$$\Phi_p^\varepsilon(t, h^*) = \rho_p - \lambda_{ij} = \rho_p + \mu_{ij} \geq \mu_{ij} \quad \forall t \in [t_0, t_f] \quad (4.39)$$

from which we conclude that

$$\mu_{ij} = \text{essinf} \{ \Phi_p^\varepsilon(t, h^*) : t \in [t_0, t_f], p \in \mathcal{P}_{ij} \} = v_{ij}(h^*) + \min_{q \in \mathcal{P}_{ij}} \{ \varepsilon_{ij}^q(h^*) \} \quad \forall (i, j) \in \mathcal{W}$$

In view of the definition of Φ_p^ε (3.15), we re-state (4.38) as

$$\begin{aligned} & h_p^*(t) > 0, p \in \mathcal{P}_{ij} \\ \implies & \max \left\{ \Psi_p(t, h^*), v_{ij}(h^*) + \varepsilon_{ij}^p(h^*) \right\} - \varepsilon_{ij}^p(h^*) + \min_{q \in \mathcal{P}_{ij}} \{ \varepsilon_{ij}^q(h^*) \} = v_{ij}(h^*) + \min_{q \in \mathcal{P}_{ij}} \{ \varepsilon_{ij}^q(h^*) \} \\ \implies & \Psi_p(t, h^*) \leq v_{ij}(h^*) + \varepsilon_{ij}^p(h^*) \quad \forall t \in [t_0, t_f], \quad \forall (i, j) \in \mathcal{W} \end{aligned}$$

which is recognized as conditions describing a dynamic user equilibrium with variable tolerance bounded rationality, where $v_{ij}(h^*)$ is the essential infimum of effective path delays between origin-destination (i, j) .

(ii) [Necessity] For each origin-destination pair $(i, j) \in \mathcal{W}$, define $\mu_{ij}^\varepsilon(h)$ to be the essential infimum of $\Phi_p^\varepsilon(\cdot, h)$ between (i, j) . It is then easy to verify that $\mu_{ij}^\varepsilon(h) = v_{ij}(h) + \min_{q \in \mathcal{P}_{ij}} \{\varepsilon_{ij}^q(h)\}$ for all $(i, j) \in \mathcal{W}$ and for all $h \in \Lambda_1$.

For a VT-BR-DUE solution h^* we have for all $(i, j) \in \mathcal{W}$ that

$$h_p^*(t) > 0, p \in \mathcal{P}_{ij} \implies v_{ij}(h^*) \leq \Psi_p(t, h^*) \leq v_{ij}(h^*) + \varepsilon_{ij}^p(h^*)$$

which, in view of (3.15), translates to

$$h_p^*(t) > 0, p \in \mathcal{P}_{ij} \implies \Phi_p^\varepsilon(t, h^*) = \mu_{ij}^\varepsilon(h^*) \quad \forall (i, j) \in \mathcal{W}$$

Therefore, given any $h \in \Lambda_1$, we have

$$\begin{aligned} \sum_{(i,j) \in \mathcal{W}} \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \Phi_p^\varepsilon(t, h^*) h_p^*(t) dt &= \sum_{(i,j) \in \mathcal{W}} \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \mu_{ij}^\varepsilon(h^*) h_p^*(t) dt \\ &= \sum_{(i,j) \in \mathcal{W}} \mu_{ij}^\varepsilon(h^*) \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p^*(t) dt \\ &= \sum_{(i,j) \in \mathcal{W}} \mu_{ij}^\varepsilon(h^*) \cdot Q_{ij} \\ &= \sum_{(i,j) \in \mathcal{W}} \mu_{ij}^\varepsilon(h^*) \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t) dt \\ &\leq \sum_{(i,j) \in \mathcal{W}} \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \Phi_p^\varepsilon(t, h^*) h_p(t) dt \end{aligned} \quad (4.40)$$

Therefore, $h^* \in \Lambda_1$ is a solution of the DVI (4.27). \square

4.1 The fixed-point reformulation and algorithm

The VT-BR-DUE also admits a fixed point problem (FPP) reformulation provided the differential variational inequality established in Theorem 4.1.

Theorem 4.2. (Fixed point problem equivalent to VT-BR-DUE) *Assume that $\Phi^\varepsilon(\cdot, h)$ defined in (3.14)-(3.15) is measurable and strictly positive for all $p \in \mathcal{P}$ and $h \in \Lambda_1$. Then the fixed point problem*

$$h^* = P_{\Lambda_1} [h^* - \alpha \Phi^\varepsilon(t, h^*)] \quad (4.41)$$

is equivalent to DVI($\Phi^\varepsilon, \Lambda_1, [t_0, t_f]$) and to BR-DUE($\Psi, \varepsilon, \Lambda_0, [t_0, t_f]$), where $P_{\Lambda_1}[\cdot]$ is the minimum norm projection onto Λ_1 and $\alpha > 0$ is a fixed constant.

Proof. The fixed point problem (4.41) requires that

$$h^* = \arg \min_h \left\{ \frac{1}{2} \|h^* - \alpha \Phi^\varepsilon(t, h^*) - h\|^2 : h \in \Lambda_1 \right\}$$

In other words, we seek a solution of the following optimal control problem

$$\min_h J(h) = \int_{t_0}^{t_f} \frac{1}{2} \sum_{(i,j) \in \mathcal{W}} \sum_{p \in \mathcal{P}_{ij}} [h^* - \alpha \Phi_p^\varepsilon(t, h^*) - h]^2 dt + \sum_{(i,j) \in \mathcal{W}} \mu_{ij} [Q_{ij} - y_{ij}(t_f)] \quad (4.42)$$

subject to

$$\frac{d}{dt} y_{ij} = \sum_{p \in \mathcal{P}_{ij}} h_p(t) \quad \forall (i, j) \in \mathcal{W} \quad (4.43)$$

$$y_{ij}(t_0) = 0 \quad \forall (i, j) \in \mathcal{W} \quad (4.44)$$

$$h \geq 0 \quad (4.45)$$

The Hamiltonian for the above optimal control problem is

$$H = \frac{1}{2} \sum_{(i,j) \in \mathcal{W}} \sum_{p \in \mathcal{P}_{ij}} [h^* - \alpha \Phi_p^\varepsilon(t, h^*) - h]^2 + \sum_{(i,j) \in \mathcal{W}} \lambda_{ij} \sum_{p \in \mathcal{P}_{ij}} h_p \quad (4.46)$$

which is convex in its controls h and has no state dependence, so the minimum principle and supporting optimality conditions are both necessary and sufficient.

The minimum principle for (4.46) has the Kuhn-Tucker conditions (the optimal solution is denoted with a double-star):

$$[h_p^* - \alpha \Phi_p^\varepsilon(t, h^*) - h_p^{**}] \cdot (-1) + \lambda_{ij} = \rho_p \quad \forall (i, j) \in \mathcal{W}, \quad p \in \mathcal{P}_{ij} \quad (4.47)$$

$$\rho_p h_p^{**} = 0 \quad \forall (i, j) \in \mathcal{W}, \quad p \in \mathcal{P}_{ij} \quad (4.48)$$

$$\rho_p \geq 0 \quad \forall (i, j) \in \mathcal{W}, \quad p \in \mathcal{P}_{ij} \quad (4.49)$$

By virtue of (4.41) and sufficiency of the minimum principle, we have that $h^* \equiv h^{**}$. Therefore, we may restate (4.47) as

$$\alpha \Phi_p^\varepsilon(t, h^*) + \lambda_{ij} = \rho_p \quad \forall (i, j) \in \mathcal{W}, \quad p \in \mathcal{P}_{ij} \quad (4.50)$$

Note that the adjoint equations and the associated transversality conditions are

$$\frac{d\lambda_{ij}}{dt} = (-1) \frac{\partial H}{\partial y_{ij}} = 0 \quad \forall (i, j) \in \mathcal{W} \quad (4.51)$$

$$\lambda_{ij}(t_f) = \frac{\partial \mu_{ij} [Q_{ij} - y_{ij}(t_f)]}{\partial y_{ij}(t_f)} = -\mu_{ij} \quad \forall (i, j) \in \mathcal{W} \quad (4.52)$$

Consequently, $\lambda_{ij}(t) \equiv -\mu_{ij}$, for all $(i, j) \in \mathcal{W}$, $t \in [t_0, t_f]$. Since $h^* \equiv h^{**}$, we have by (4.48) that $h_p^* > 0$ implies $\rho_p = 0$, $\forall p \in \mathcal{P}$. Hence

$$h_p^*(t) > 0, p \in \mathcal{P}_{ij} \implies \alpha \Phi_p^\varepsilon(t, h^*) = -\lambda_{ij} = \mu_{ij} \implies \Phi_p^\varepsilon(t, h^*) = \frac{\mu_{ij}}{\alpha} \doteq v_{ij} \quad \forall \nu(t) \in [t_0, t_f]$$

Furthermore, (4.49) and (4.50) imply that $\Phi_p^\varepsilon(t, h^*) \geq \frac{\mu_{ij}}{\alpha} = v_{ij}$, $\forall \nu t \in [t_0, t_f]$, that is, v_{ij} is the essential infimum of $\Phi^\varepsilon(t, h^*)$ corresponding to $(i, j) \in \mathcal{W}$. Using the same deductions as shown in (4.40), we conclude that h^* is a solution of the DVI problem (4.27). \square

A direct benefit of invoking the DVI and the fixed point problem formulations is that it is possible to develop a computational scheme based on the fixed point iterations. Namely, Theorem 4.2 suggests the algorithm

$$h^{k+1} = P_{\Lambda_1} \left[h^k - \alpha \Phi^\varepsilon(t, h^k) \right]$$

That is, at the k -th iteration, one needs to solve the following linear-quadratic optimal control problem

$$h^{k+1} = \arg \min_h \left\{ \frac{1}{2} \left\| h^k - \alpha \Phi^\varepsilon(t, h^k) - h \right\|^2 : h \in \Lambda_1 \right\}$$

That is, we seek the solution of the optimal control problem

$$\min_h J^k(h) = \sum_{(i,j) \in \mathcal{W}} \mu_{ij} [Q_{ij} - y_{ij}(t_f)] + \sum_{(i,j) \in \mathcal{W}} \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \frac{1}{2} \left[h_p^k - \alpha \Phi_p^\varepsilon(t, h) - h_p \right]^2 \quad (4.53)$$

subject to:

$$\frac{dy_{ij}}{dt} = \sum_{p \in \mathcal{P}_{ij}} h_p(t) \quad \forall (i, j) \in \mathcal{W} \quad (4.54)$$

$$y_{ij}(t_0) = 0 \quad \forall (i, j) \in \mathcal{W} \quad (4.55)$$

$$h \geq 0 \quad (4.56)$$

Finding dual variables associated with terminal time demand constraints turns out to be relatively easy. Note that the relevant Hamiltonian for (4.53), (4.54), (4.55) and (4.56) is:

$$H^k = \frac{1}{2} \sum_{(i,j) \in \mathcal{W}} \sum_{p \in \mathcal{P}_{ij}} \left[h_p^k - \alpha \Phi_p^\varepsilon(t, h^k) - h_p \right]^2 + \sum_{(i,j) \in \mathcal{W}} \lambda_{ij} \sum_{p \in \mathcal{P}_{ij}} h_p \quad (4.57)$$

where each λ_{ij} is an adjoint variable obeying

$$\begin{aligned} \frac{d\lambda_{ij}}{dt} &= (-1) \frac{\partial H^k}{\partial y_{ij}} = 0 \quad \forall (i, j) \in \mathcal{W} \\ \lambda_{ij}(t_f) &= \frac{\partial \mu_{ij} [Q_{ij} - y_{ij}(t_f)]}{\partial y_{ij}(t_f)} = -\mu_{ij} \quad \forall (i, j) \in \mathcal{W} \end{aligned}$$

From the above we determine that

$$\lambda_{ij}(t) \equiv -\mu_{ij} \quad \forall t \in [t_0, t_f], \quad (i, j) \in \mathcal{W}$$

The minimum principle implies for any $p \in \mathcal{P}$,

$$h_p^{k+1}(t) = \arg \left\{ \frac{\partial H^k}{\partial h_p} = 0 \right\} = \arg \left\{ \left[\left(h_p^k(t) - \alpha \Phi_p^\varepsilon(t, h^k) - h_p(t) \right) \right] (-1) - \mu_{ij} = 0 \right\}$$

Thus, we obtain

$$h_p^{k+1}(t) = \left[h_p^k(t) - \alpha \Phi_p^\varepsilon(t, h^k) + \mu_{ij} \right]_+ \quad \forall (i, j) \in \mathcal{W}, \quad p \in \mathcal{P}_{ij} \quad (4.58)$$

Notice that the following flow conservation constraint applies here.

$$\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p^{k+1}(t) dt = Q_{ij} \quad \forall (i, j) \in \mathcal{W}$$

Consequently the dual variable μ_{ij} must satisfy

$$\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \left[h_p^k(t) - \alpha \Phi_p^\varepsilon(t, h^k) + \mu_{ij} \right]_+ dt = Q_{ij} \quad \forall (i, j) \in \mathcal{W} \quad (4.59)$$

Recall that each μ_{ij} is time invariant, and that equations (4.59) are uncoupled. Furthermore, the left hand side of (4.59) is strictly increasing in μ_{ij} . Thus, a simple root search algorithm will identify the values of μ_{ij} according to (4.59). Once the μ_{ij} 's are determined, the new vector of path flows h^{k+1} for the next iteration may be computed from (4.58). As a result, we have the following algorithm for solving both the VT-BR-DUE and BR-DUE problems.

Fixed-point Algorithm for SRDC VT-BR-DUE and BR-DUE

Step 0. Initialization. Identify an initial feasible solution $h^0 \in \Lambda_1$ and set iteration counter $k = 0$

Step 1. Find Φ^ε . Solve the dynamic network loading subproblem with path flows given by h^k , and obtain the effective path delays $\Psi(\cdot, h^k)$. Then define, for each $p \in \mathcal{P}_{ij}$, the following quantities

$$\Phi_p^\varepsilon(t, h^k) = \max \left\{ \Psi_p(t, h^k), v_{ij}(h^k) + \varepsilon_{ij}^p(h^k) \right\} - \left(\varepsilon_{ij}^p(h^k) - \min_{q \in \mathcal{P}_{ij}} \left\{ \varepsilon_{ij}^q(h^k) \right\} \right) \quad (\text{VT-BR-DUE})$$

or

$$\Phi_p^\varepsilon(t, h^k) \doteq \max \left\{ \Psi_p(t, h^k), v_{ij}(h^k) + \varepsilon_{ij} \right\} \quad (\text{BR-DUE})$$

where $v_{ij}(h^k)$ is the minimum effective delay between origin-destination pair (i, j) .

Step 2. Find the dual variable. For each $(i, j) \in \mathcal{W}$, solve the following equation for μ_{ij} , using root-search algorithms.

$$\sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_f} \left[h_p^k(t) - \alpha \Phi_p^\varepsilon(t, h^k) + \mu_{ij} \right]_+ dt = Q_{ij}$$

Step 3. Update path flows. For each $(i, j) \in \mathcal{W}$ and $p \in \mathcal{P}_{ij}$, update the path flow as

$$h_p^{k+1}(t) = \left[h_p^k(t) - \alpha \Phi_p^\varepsilon(t, h^k) + \mu_{ij} \right]_+ \quad \forall t \in [t_0, t_f]$$

Step 4. Check for convergence. Terminate algorithm with output $h^* \approx h^k$ if

$$\frac{\|h^{k+1} - h^k\|_{L^2}}{\|h^k\|_{L^2}} \leq \epsilon$$

where $\epsilon \in \mathbb{R}_{++}$ is a prescribed termination threshold, and

$$\|h\|_{L^2} \doteq \sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} h_p^2(t) dt$$

is the standard L^2 -norm on $(L^2[t_0, t_f])^{|\mathcal{P}|}$. Otherwise, set $k = k + 1$ and repeat Step 1 through Step 4.

Remark 4.3. *Convergence of the proposed fixed point algorithm, as in all other algorithms, depends on properties of the principle operator, that is, $\Phi^\varepsilon(t, h)$. As pointed out by Friesz et al. (2011), a sufficient condition for convergence of the fixed point algorithm is strong monotonicity together with Lipschitz continuity of such operator. While this condition might be quite restrictive, there exists a number of other computational methods based on the proposed VI formulation that allow weaker convergence conditions to be imposed, such as the descent direction method (Han and Lo, 2003; Szeto and Lo, 2004), and the projection method (Ukkusuri et al., 2012). In any case, a central focus of investigation is the property of the newly introduced operator Φ^ε defined in (3.14)-(3.15). This direction of research is beyond the scope of this paper and will be pursued in the future.*

5 Existence of VT-BR-DUE and BR-DUE

The existence of VT-BR-DUE or BR-DUE are most easily analyzed through the established VI formulations (Theorem 3.1, Lemma 3.2) and Brouwer’s fixed-point existence theorem. One statement of Brouwer’s theorem appears as Theorem 2 of Browder (1968):

Theorem 5.1. (Browder, 1968) *Let K be a compact convex subset of the locally convex topological vector space E , T a continuous (single-valued) mapping of K into E^* , the dual space of E . Then there exists u_0 in K such that*

$$\left\langle T(u_0), u_0 - u \right\rangle \geq 0 \quad \forall u \in K$$

Proof. See Browder (1968). □

Approaches based on Brouwer’s theorem require the set of feasible path flows (departure rates) under consideration to be compact and convex in a Hilbert space, and typically involve an *a priori* bound on all the path flows (Zhu and Marcotte, 2000). The invocation of the *a priori* boundedness on path flows is used to secure compactness needed for a topological argument; the reason is that, in contrast to the finite-dimensional Euclidean spaces, closedness and boundedness together no longer guarantee compactness of a subset in infinite-dimensional function spaces.

Notes should be taken on the following fact: the existence of a dynamic user equilibrium always trivially implies the existence of a bounded rationality DUE. Therefore, a meaningful discussion of the existence of VT-BR-DUEs or BR-DUEs should rely on assumptions weaker than those made for DUEs. This is precisely the aim of this section. In particular, we will first review the existence result for simultaneous route-and-departure choice (SRDC) DUE established by Han et al. (2013c), which is arguably the most general existence result in the literature. In fact, as will be shown later, a violation of the condition stipulated in Han et al. (2013c) will result in the non-existence of SRDC DUEs. In the meantime, we will provide the existence results for VT-BR-DUE with conditions weaker than those assumed by Han et al. (2013c) for the SRDC DUE case, and therefore claiming a more general existence result for VT-BR-DUE.

5.1 Review of existence result for SRDC DUE

Han et al. (2013c) use the same set of notations as presented in Section 2.1 to establish the existence of simultaneous route-and-departure choice dynamic user equilibrium. In that paper,

the effective path travel delay is expressed as the sum of two functions, i.e., for any $(i, j) \in \mathcal{W}$,

$$\Psi_p(t, h) = \phi_{ij}(t) + \psi_{ij}(\tau_p(t)) \quad \forall p \in \mathcal{P}_{ij}$$

where t denotes the departure time, and $\tau_p(t)$ denotes the path exit time corresponding to departure time t . The specific form of effective path delay given in (2.1) is now instantiated with

$$\phi_{ij}(t) = -t, \quad \psi_{ij}(\tau_p(t)) = \tau_p(t) + f(\tau_p(t) - T_A)$$

The invocation of functions $\phi_{ij}(\cdot)$ and $\psi_{ij}(\cdot)$ is for the ease of analysis. The combination of the two functions can represent rather general travel cost functions including not only those of the form (2.1), but also the following cost function frequently employed in the DTA literature.

$$\alpha(\tau_p(t) - t) + \begin{cases} \beta(T_A - \tau_p(t)) & \tau_p(t) \leq T_A \\ \gamma(\tau_p(t) - T_A) & \tau_p(t) > T_A \end{cases} \quad (5.60)$$

where $\gamma > \alpha > \beta > 0$. The reader is referred to Han et al. (2013c) for a more detailed demonstration.

Han et al. (2013c) put forward the following sufficient conditions for the existence of SRDC DUE.

A1. For each $(i, j) \in \mathcal{W}$, $\phi_{ij}(\cdot)$ and $\psi_{ij}(\cdot)$ are continuous on $[t_0, t_f]$. Moreover, $\phi_{ij}(\cdot)$ is monotonically decreasing while $\psi_{ij}(\cdot)$ is monotonically increasing. In addition, it is assumed that $\phi_{ij}(\cdot)$ is Lipschitz continuous with constant L_{ij} ; and there exists $\Delta_{ij} > 0$ such that

$$\psi_{ij}(t_2) - \psi_{ij}(t_1) \geq \Delta_{ij}(t_2 - t_1) \quad \forall t_0 \leq t_1 < t_2 \leq t_f \quad (5.61)$$

A2. Each link $a \in \mathcal{A}$ of the network has a finite exit flow capacity $M_a < \infty$.

A3. The effective delay operator Ψ is continuous from Λ_0 into $(L_+^2[t_0, t_f])^{|\mathcal{P}|}$.

Remark 5.2. Notice that assumption **A1** still holds true for the cost function (5.60) with $\gamma > \alpha > \beta > 0$. A proof is provided in Han et al. (2013c). Assumption **A2** applies to a large portion of traffic flow models with only a few exceptions, including the link delay model proposed by Friesz et al. (1993).

Assumption **A3** is the most recognized condition for existence, and it has been shown to be true when **A3** is associated with the link delay model (Han et al., 2012) and with the Vickrey model (Han et al., 2013c). Assumptions **A1-A2** together are meant to tackle the issue of compactness mentioned earlier, which would otherwise be treated using the *ad hoc* boundedness on path flows.

We will next proceed with our discussion on the existence of BR-DUE according to the following structure. We will first illustrate the non-existence of SRDC DUE without assumptions **A1** and **A2**, when the link delay model and the Vickrey model are respectively considered. Then we will show the existence of VT-BR-DUE and BR-DUE in the absence of **A1** and **A2**, thus claiming a more general existence result for DUEs with bounded rationality.

5.2 Non-existence of SRDC DUE in violation of A1 or A2

5.2.1 Non-existence of SRDC DUE with the Vickrey model

Bressan and Han (2011) show the existence and uniqueness of a Nash-like traffic equilibrium on a network with one link and a single bottleneck. That paper employs a hydrodynamic model

with a point-queue at the traffic bottleneck, of which the Vickrey model (Vickrey, 1969) is a special case. A numerical example of such equilibrium is provided based on the following choices of travel cost functions

$$\phi(t) = -t, \quad \psi(\tau_p(t)) = \begin{cases} 0 & \tau_p(t) \leq T_A \\ (\tau_p(t) - T_A)^2 & \tau_p(t) > T_A \end{cases}$$

where T_A is the target arrival time, t denotes departure time, and $\tau_p(t)$ denotes path exit time. The subscript ij is omitted from the cost functions ϕ and ψ since there is only one origin-destination pair in the network. Clearly, the definition of ψ violates assumption **A1**, or (5.61), to be precise. Bressan and Han (2011) provide a closed-form characterization of the traffic equilibrium, for which the path departure rate is not a square-integrable function; rather, it is a distribution. Since such a distribution is the only solution for this traffic equilibrium problem, we conclude that a dynamic user equilibrium defined in the sense of Friesz et al. (1993) (i.e., using the square-integrable function space) does not exist. This is directly due to the violation of **A1**.

5.2.2 Non-existence of SRDC DUE with the link delay model

The link delay model (LDM) is originally proposed by Friesz et al. (1993). It belongs to the class of explicit travel time function models. In particular, it is assumed that the link traversal time, denoted by $D_a(t)$, is expressed as an affine function of the link occupancy $X(t)$ at the time of entry, where t denotes the link entry time:

$$D_a(t) = \alpha X(t) + \beta$$

for some $\alpha, \beta > 0$. Clearly, the LDM allows arbitrary link exit flow – a violation of assumption **A2**. As a result, the existence of SRDC DUE with LDM cannot be established in the framework proposed by Han et al. (2013c). When this happens, one may rely on the restrictive assumption that all the path flows are *a priori* bounded from above in order to prove existence (Zhu and Marcotte, 2000), but that proof does not guarantee the existence of any DUE solution characterized solely by the set Λ_0 , which contains unbounded functions.

5.3 Existence result for VT-BR-DUE and BR-DUE

Our main existence result for the VT-BR-DUE problem, and also for the BR-DUE problem as a special case, is stated as follows.

Theorem 5.3. (Existence of VT-BR-DUE) *Assume that*

- (1) *The effective delay operator $\Psi: \Lambda_0 \rightarrow (L_+^2[t_0, t_f])^{|\mathcal{P}|}$ is continuous.*
- (2) *The effective path delays $\Psi_p(t, h)$ is Lipschitz continuous with respect to t , $\forall h \in \Lambda_0$.*
- (3) *The functionals $\varepsilon_{ij}^p(\cdot): \Lambda_0 \rightarrow \mathbb{R}_+$ is bounded away from zero. That is, there exists $\varepsilon^{\min} > 0$ such that $\varepsilon_{ij}^p(h) \geq \varepsilon^{\min}$ for all $h \in \Lambda_0$.*

Then the variable tolerance bounded rationality dynamic user equilibrium with simultaneous route-and-departure time choice, as defined in Definition 2.3, exists.

Proof. We consider, for each natural number $n \geq 1$, a uniform partition of the compact interval $[t_0, t_f]$ into n sub-intervals I_1, \dots, I_n with the size of each being $(t_f - t_0)/n$. We then consider the following finite-dimensional subsets

$$\Lambda^n \doteq \{h \in \Lambda_0 : h_p(\cdot) \text{ is constant on } I_i, \quad \forall p \in \mathcal{P}\} \subset \Lambda_0 \quad \forall n \geq 1$$

Note that each Λ^n is the intersection of Λ_0 and the space of piecewise constant functions on I_n . Since Λ_0 is expressed via linear constraints, each set Λ^n is clearly convex. In addition, due to the finite-dimensional nature of Λ^n , we conclude that it is also compact.

It now follows from assumption (1) and Theorem 5.1 that for each $n \geq 1$, there exists $h^{n,*} \in \Lambda^n$ such that

$$\langle \Psi(t, h^{n,*}), h^n - h^{n,*} \rangle \geq 0 \quad \text{or} \quad \sum_{p \in \mathcal{P}} \int_{t_0}^{t_f} \Psi_p(t, h^{n,*}) (h_p^n(t) - h_p^{n,*}(t)) dt \geq 0, \quad \forall h^n \in \Lambda^n \quad (5.62)$$

Since both h^n and $h^{n,*}$ are both piecewise constant, it follows from (5.62) that for any $(i, j) \in \mathcal{W}$,

$$h_p^{n,*}(t) > 0, \quad t \in I_k \implies \int_{I_k} \Psi_p(t, h^{n,*}) dt = \min_{q \in \mathcal{P}_{ij}} \min_{l=1, \dots, n} \int_{I_l} \Psi_q(t, h^{n,*}) dt \quad (5.63)$$

for all $p \in \mathcal{P}_{ij}$ and $k = 1, \dots, n$. In other words, within an origin-destination pair, the integrations of the effective path delay for all utilized paths and departure time intervals are equal and minimal.

In view of assumption (3), we choose $n \geq 1$ such that the size of the subintervals $\delta_n \doteq \frac{t_0 - t_f}{n}$ satisfies $\delta_n < \frac{\varepsilon^{min}}{2L}$. Fix any origin-destination pair $(i, j) \in \mathcal{W}$, we denote by $v_{ij}(h^{n,*})$ the essential infimum of the path effective delays, which it is attained in some subinterval I_l corresponding to some path $q \in \mathcal{P}_{ij}$.

We claim that for any $p \in \mathcal{P}_{ij}$ and any $t \in [t_0, t_f]$,

$$h_p^{n,*}(t) > 0 \implies \Psi_p(t, h^{n,*}) \leq v_{ij}(h^{n,*}) + \varepsilon^{min} \quad (5.64)$$

Otherwise, by contradiction we assume that there exists some path $p \in \mathcal{P}_{ij}$ and time interval I_k such that

$$h_p^{n,*}(t) > 0, \quad t \in I_k, \quad \Psi_p(\bar{t}, h^{n,*}) > v_{ij}(h^{n,*}) + \varepsilon^{min} \quad (5.65)$$

for some $\bar{t} \in I_k$. According to assumption (2), we denote by L the Lipschitz constant of $\Psi_p(\cdot, h^{n,*})$. Then we have the following inequality

$$\begin{aligned} \left| \frac{\Psi_p(t, h^{n,*}) - \Psi_p(\bar{t}, h^{n,*})}{\delta_n} \right| &\leq \left| \frac{\Psi_p(t, h^{n,*}) - \Psi_p(\bar{t}, h^{n,*})}{t - \bar{t}} \right| \leq L \\ \implies \Psi_p(t, h^{n,*}) &\geq \Psi_p(\bar{t}, h^{n,*}) - \delta_n L \quad \forall t \in I_k \end{aligned} \quad (5.66)$$

A similar argument is carried out in the subinterval I_l corresponding to the path q , and we have that

$$\Psi_q(t', h^{n,*}) \leq v_{ij}(h^{n,*}) + \delta_n L \quad \forall t' \in I_l \quad (5.67)$$

(5.65), (5.66) and (5.67) together imply that

$$\Psi_p(t, h^{n,*}) \geq v_{ij}(h^{n,*}) + \varepsilon^{min} - \delta_n L \geq \Psi_q(t', h^{n,*}) - 2\delta_n L + \varepsilon^{min} > \Psi_q(t', h^{n,*}) \quad (5.68)$$

for all $t \in I_k, t' \in I_l$. We thus conclude that

$$\int_{I_k} \Psi_p(t, h^{n,*}) dt > \int_{I_l} \Psi_q(t', h^{n,*}) dt'$$

which yields contradiction to (5.63). Thus, the established (5.64) shows that $h^{n,*}$ leads to a VT-BR-DUE. \square

6 Numerical studies

This section numerically illustrates the performances of the proposed fixed-point algorithm for computing VT-BR-DUEs and BR-DUEs, in terms of solution features, number of iterations, convergence and computational time. We will consider several test networks of varying sizes. For the dynamic network loading procedure, we employ the Lighthill-Whitham-Richards kinematic wave model (Lighthill and Whitham, 1955; Richards, 1956), with the network extension that allows vehicle spillback to be explicitly captured (Garavello and Piccoli, 2006; Han et al., 2013d). We note that the fixed-point algorithm accommodates other types of network loading procedures as well.

All of the reported computations of the fixed-point algorithms are performed on a computer with 2.7 GHz processor and 4 GB RAM. Every attempt was made to avoid the use of numerical tricks that could not be equally applied to all models.

6.1 The seven-arc, six-node network

Our first network example is meant to illustrate the VT-BR-DUE model in which the user tolerances are endogenous, that is, they depend on the realized path flows. We use this example also to demonstrate the ‘hot route’ mentioned earlier in the introduction, for which the cost tolerance depends on the traffic volume on that route. In particular, we will show how such dependence affects final outcome of the VT-BR-DUE solution.

The test network is illustrated in Figure 3, where there is one origin-destination pair (1, 6) and three paths $p_1 = \{a, c, e, g\}$, $p_2 = \{a, b, d, e, g\}$, $p_3 = \{a, b, f, g\}$.

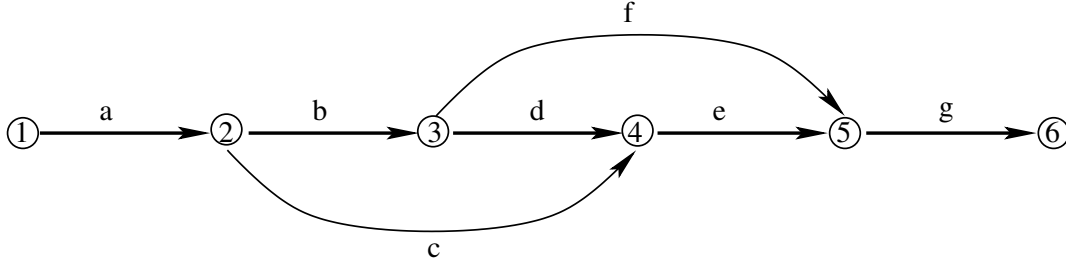


Figure 3: The seven-arc, six-node test network

The fixed travel demand is 2000 vehicles. We regard path p_2 as the hot route, and select the tolerances in the following fashion, in two cases.

$$\text{Case I. } \varepsilon_1 = \varepsilon_3 \equiv 0.1, \varepsilon_2 = 0.15 \left(1 - \frac{100}{100 + V_2} \right)$$

$$\text{Case II. } \varepsilon_1 = \varepsilon_3 \equiv 0.1, \varepsilon_2 = 0.2 \left(1 - \frac{100}{100 + V_2} \right)$$

where ε_i denotes the user tolerance associated with path p_i , $i = 1, 2, 3$; and $V_2 \doteq \int_{t_0}^{t_f} h_{p_2}(t) dt$ is the total traffic volume on path p_2 . In both cases, the cost tolerance increases with the traffic volume V_2 , and is bounded from above by a fixed constant; see Figure 4 for a visualization of such dependence. Notice that *Case II* gives a higher tolerance along p_2 than *Case I* provided the same value of V_2 . By comparing the two cases, we will show how the slight difference in the tolerance functions manifest itself in the solution. We further note that the functional forms for the tolerances chosen above are for illustration purposes only. Study is needed to formulate and calibrate those specific functional forms but will not be discussed in this paper.

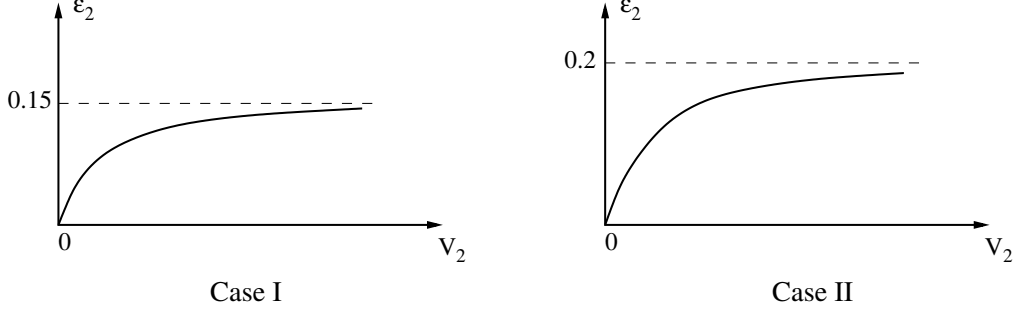


Figure 4: The VT-BR-DUE problem: Functional forms selected for ε_2 .

The VT-BR-DUE problems are solved with the proposed fixed-point algorithm and displayed in Figure 5, where we show the departure rates (path flows) along the three paths and the corresponding effective delays. Notice that the resulting volumes on path p_2 are

$$\text{Case I: } V_2 = 1105, \quad \text{Case II: } V_2 = 1178$$

Therefore the corresponding tolerances are:

$$\text{Case I. } \varepsilon_1 = \varepsilon_3 = 0.1, \varepsilon_2 = 0.1376$$

$$\text{Case II. } \varepsilon_1 = \varepsilon_3 = 0.1, \varepsilon_2 = 0.1844$$

The minimum effective delay v_{16} , and the tolerance thresholds $v_{16} + \varepsilon_i$, $i = 1, 2, 3$, are shown in Figure 5, from which we see that the computed solutions are indeed solutions of the VT-BR-DUE problems since $h_{p_i}^*(t) > 0$ implies that $\Psi(t, h_{p_i}^*) \leq v_{16} + \varepsilon_i$, $i = 1, 2, 3$. We also observe that as a result of the chosen forms of $\varepsilon_2(\cdot)$, the total traffic volume on path p_2 in *Case I* is smaller than that in *Case II*. This is because, given the same traffic volume V_2 , drivers following path p_2 in *Case I* have a lower tolerance than that in case *Case II*, thus more driver will switch to the other paths in this case.

Finally, for both computational scenarios the fixed-point algorithm converges after finite number of iterations. This can be seen from Figure 6, where the relative gap is expressed by

$$\frac{\|h^{k+1} - h^k\|_{L^2}}{\|h^k\|_{L^2}} \quad (6.69)$$

and the L^2 -norm is defined in (2.4).

6.2 The 19-arc, 13-node network

Purpose of this numerical test is to evaluate the impact of ε on the final solution of the BR-DUE problem. Our numerical result demonstrates sensitivity of the BR-DUE solution on the choices of tolerances; it also highlights the importance of a well-calibrated indifference band in applying this type of models.

The second test network is shown in Figure 7, with four origin-destination pair (1, 2), (1, 3), (4, 2), (4, 3). There are totally 24 paths among these four O-D pairs. The travel demand is 2000 vehicles for each of the four O-D pairs. We employ a fixed tolerance ε for all the O-D pairs in our computation. We consider two scenarios: (1) $\varepsilon = 0.4$, and (2) $\varepsilon = 0.2$. The corresponding BR-DUE solutions are partially illustrated in Figure 8 and Figure 9, respectively. Notice that the effective delays in all the figures refer to the functions

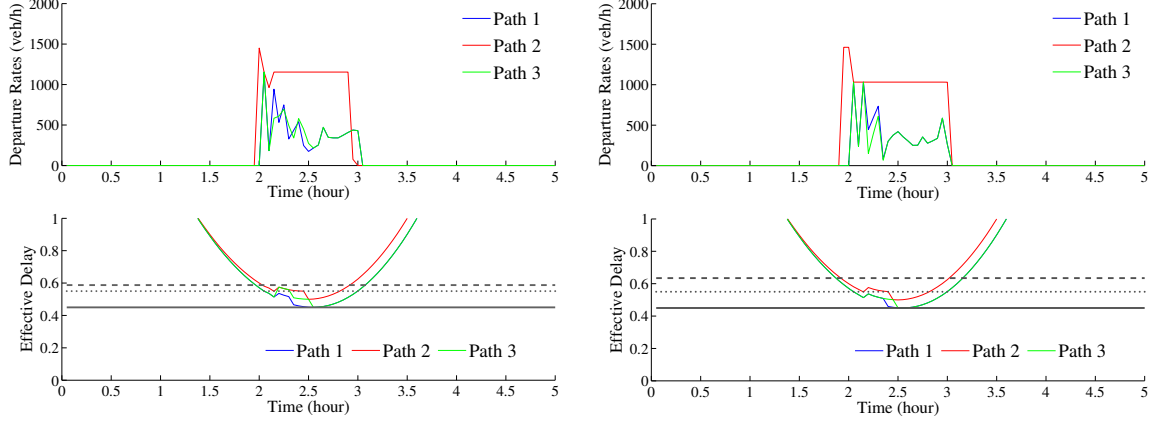


Figure 5: Solutions of the VT-BR-DUE problems in *Case I* (left) and *Case II* (right). In the figures showing the effective delays, the solid horizontal line represents the minimum effective delay v_{16} , the dashed lines represents $v_{16} + \varepsilon_2$, and the dotted lines represent $v_{16} + \varepsilon_1$ (or $v_{16} + \varepsilon_3$).

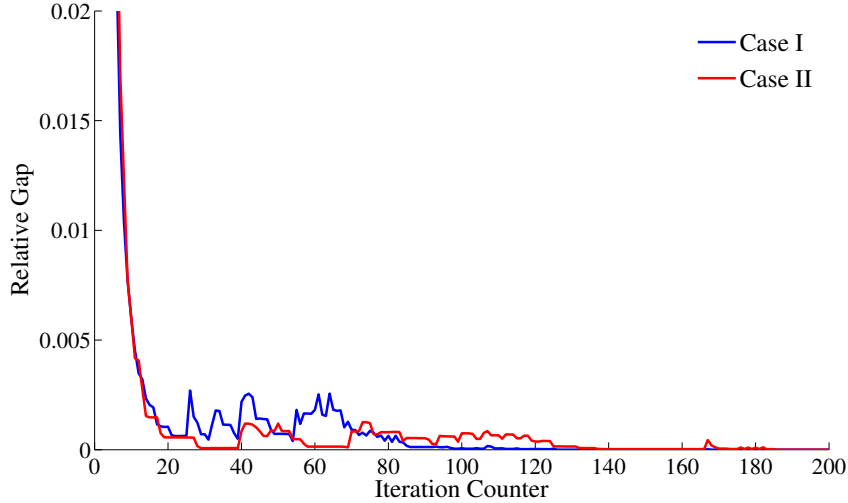


Figure 6: Convergence of the fixed-point algorithm for the seven-arc, six-node network. The relative gap is defined in (6.69).

$\phi_p^\varepsilon(t, h^*)$, which are defined in (3.22). Recall that a BR-DUE solution h^* must solve the following variational inequality

$$\langle \phi^\varepsilon(t, h^*), h - h^* \rangle \geq 0 \quad \forall h \in \Lambda_0$$

According to our earlier discussion following Lemma 3.2, we are assured that the solutions presented in Figure 8 and Figure 9 are indeed BR-DUE solutions, since $h_p^*(t) > 0$ implies that $\phi_p^\varepsilon(t, h^*)$ is equal and minimal.

We also see from a comparison between Figure 8 and Figure 9 that different choices of ε lead to very different BR-DUE solutions. Therefore, it is crucial for the application of the BR-DUE models to identify appropriate values of the cost tolerance. In addition, as we shall demonstrate later in Section 6.3, the larger the value of ε , the fewer iterations the fixed-point algorithm takes to converge.

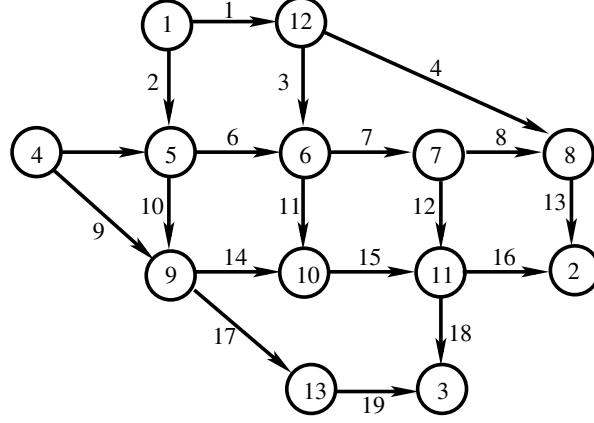


Figure 7: The 19-arc, 13-node test network

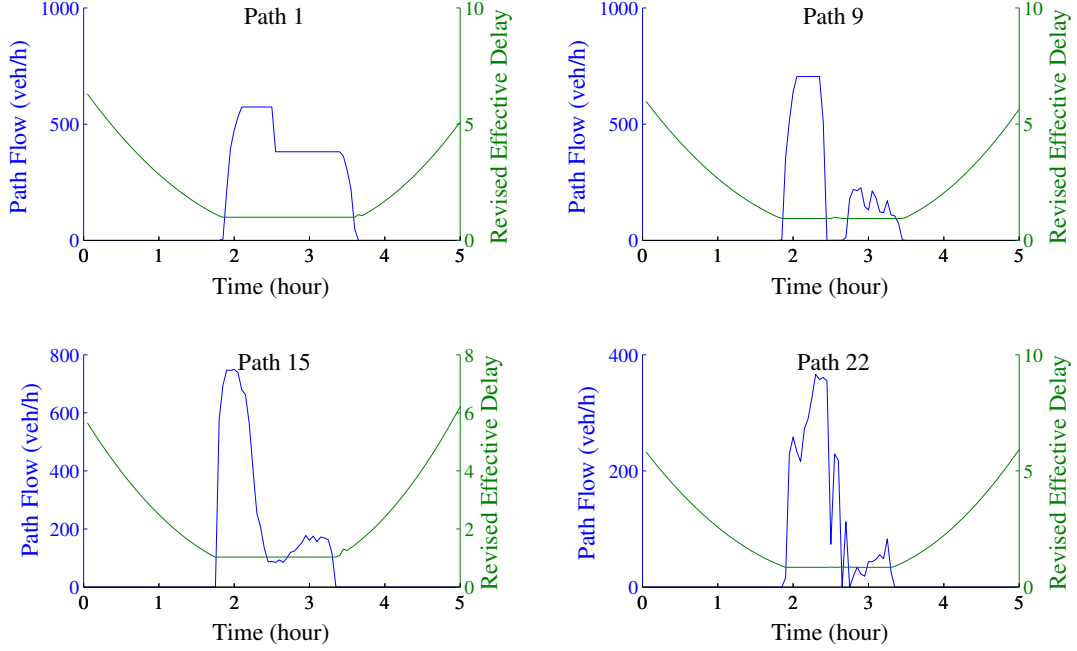


Figure 8: Scenario (1), $\varepsilon = 0.4$: BR-DUE path flows and the corresponding revised effective path delays ϕ^ε .

6.3 The Sioux Falls network

Our final test network is the 76-arc, 24-node Sioux Falls network illustrated in Figure 10. We select six origin-destination pairs: (1, 20), (2, 20), (3, 20), (4, 20), (5, 20) and (6, 20), among which 119 paths are selected.

The main purpose of this subsection is to evaluate performance of the proposed fixed-point algorithm in terms of convergence. To make our numerical study more comprehensive, we also include the previously studied seven-arc network and 19-arc network. In particular, we will compare the convergence results with varying values of ε . To simplify our problem, we employ the same fixed tolerance ε for all the O-D pairs and paths in each computational instance of the BR-DUE problem. The convergence results of the fixed-point algorithm implemented on

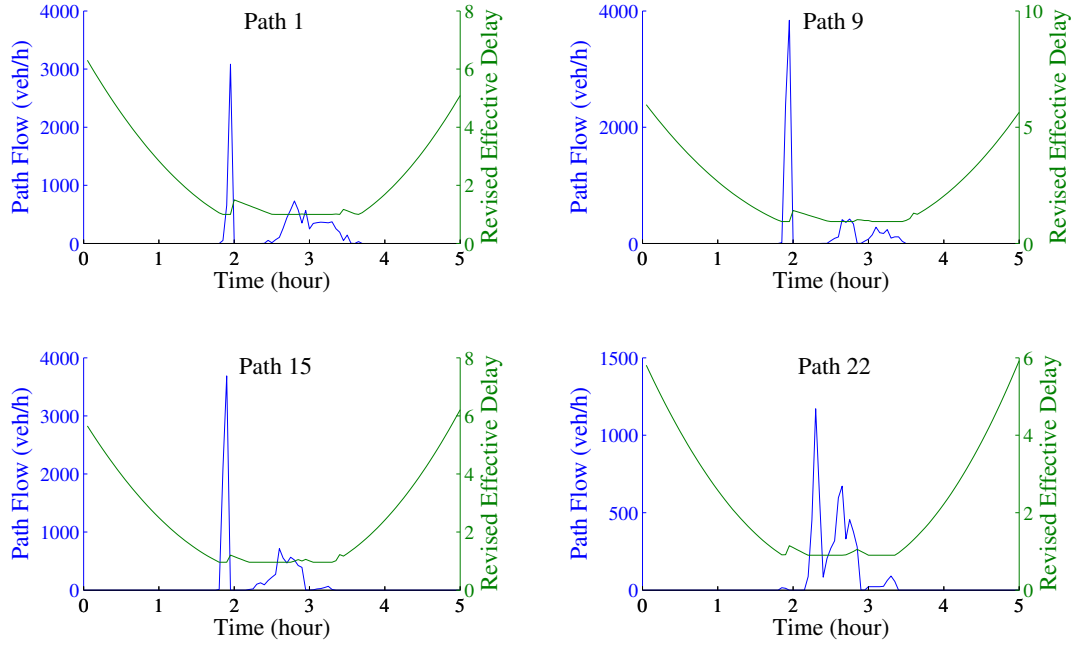


Figure 9: Scenario (2), $\varepsilon = 0.2$: BR-DUE path flows and corresponding revised effective path delays ϕ^ε .

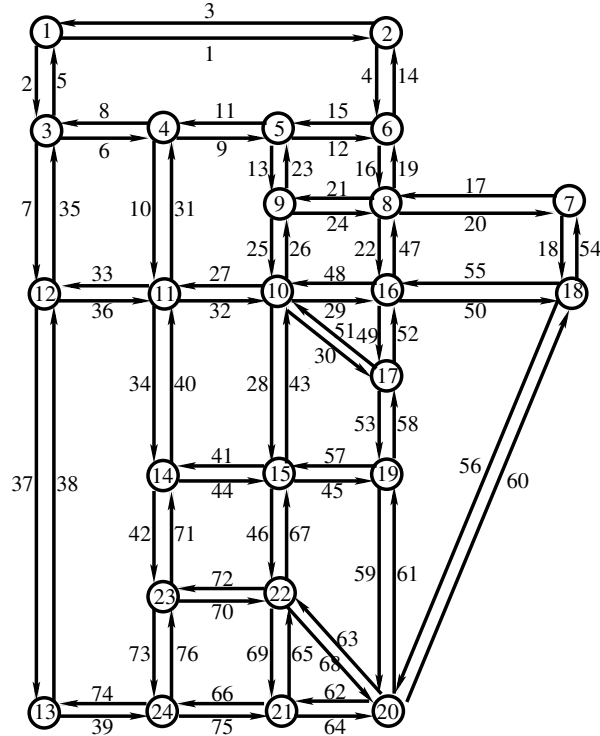


Figure 10: The Sioux Falls network

different networks are shown graphically in Figure 11, where we consider a range of values of ε . Note that for all the calculations the termination threshold of the algorithm is 10^{-4} .

The results show the following trend: as the tolerance ε becomes larger, the algorithm takes fewer iterations to converge. This is consistent with the intuitive observation that the BR-DUE solution set grows bigger as the tolerance ε increases, and therefore a sequence of points generated by the fixed-point algorithm is more likely to converge. We also see from the figure that when the test network becomes bigger, despite the increase in problem dimension (the number of paths), the number of iterations required by the fixed-point algorithm remains roughly the same, which highlights the scalability of our proposed computational method. The computational time, on the other hand, is proportional to the number of iterations; this is because the algorithm performs the dynamic network loading repeatedly and each DNL procedure consumes equal amount of time for the same network.

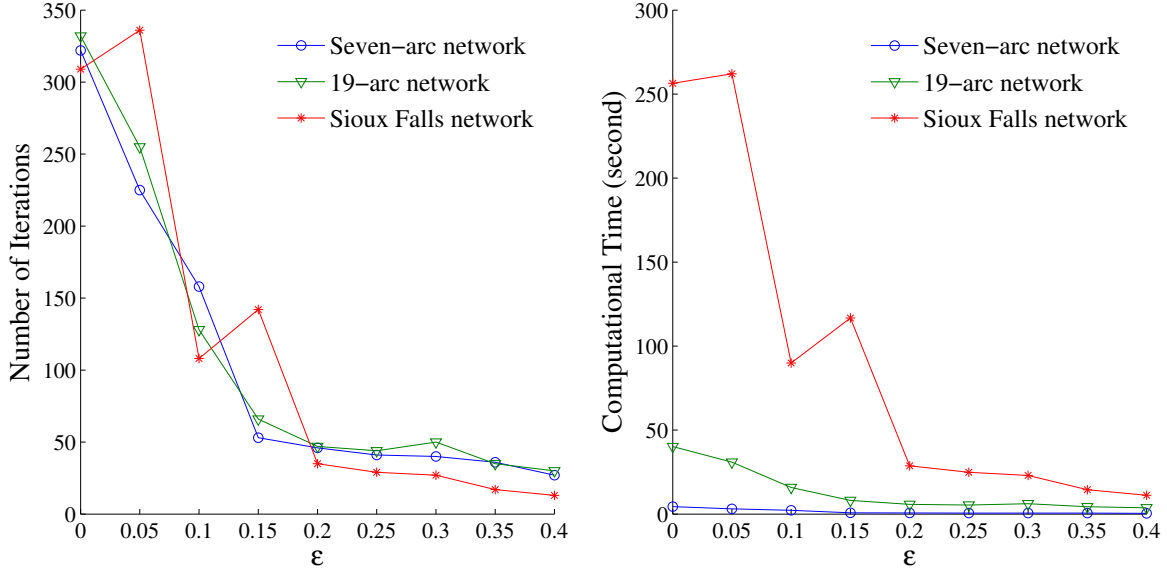


Figure 11: Number of fixed-point iterations (left) and computational time (right) required by the fixed-point algorithm, when different values of ε are chosen. Notice that when $\varepsilon = 0$, the problem reduces to a dynamic user equilibrium. For all the calculations the algorithm termination criterion is $RG \leq 10^{-4}$ where RG , the relative gap, is defined in (6.69).

7 Concluding remarks

This paper presents the first set of analytical results regarding the simultaneous route-and-departure dynamic user equilibrium with bounded rationality (BR-DUE). Specifically, we consider the BR-DUE problem with exogenously given tolerances or endogenously determined tolerances. The former problem is referred to as BR-DUE, while the latter is named VT-BR-DUE where VT stands for ‘variable tolerance’. The VT-BR-DUE problem is an extension of the BR-DUE problem; justification of such extension is elaborated in the introductory part of this paper. Both BR-DUE and VT-BR-DUE with simultaneous route-and-departure choice are formulated in this paper as variational inequalities (VI), differential variational inequalities (DVI), and fixed point problems (FPP) in an infinite-dimensional Hilbert space. It is significant that the proposed formulations can capture realistic travel behaviors, i.e., those taking into account path heterogeneity and prevailing traffic condition, by incorporating these

factors into a variable tolerance vector, and remain mathematically canonical and coherent. The three equivalent formulations for VT-BR-DUE admit rigorous mathematical investigation of qualitative properties of bounded rationality DUE, and allow a wide ranges of computational methods developed for VI, DVI and FPP to be applied for solving VT-BR-DUE problems.

We show, using measure-theoretic argument, that the VT-BR-DUE problem is equivalent to a VI problem defined in a Hilbert space. The key ingredient of the proof is a revised definition of the principle operator of the VI; the new VI can handle not only drivers' tolerances distinguished by path, but also tolerances that may depend on the prevailing traffic condition. Existence of VT-BR-DUE is rigorously analyzed under conditions weaker than those ensuring existence of SRDC DUE. We further show that the VT-BR-DUE problem admits DVI and FPP representations, when the minimum principle for the optimal control problem is invoked. Computability of the VT-BR-DUE models is demonstrated with a fixed-point algorithms developed through the FPP formulation, which is tested on several networks with satisfactory solution quality and convergence results shown.

Although this paper is mainly concerned with dynamic modeling, the techniques used to convert a BR-DUE problem into a generic VI form is applicable to static problems, i.e., BR-UE (bounded rationality user equilibrium). In addition, the route-choice dynamic user equilibrium with bounded rationality can be treated in a very similar way; and it is expected that VI formulation, existence result, and computational methods will become available for this type of problems as well. Due to space limitation, those results are not elaborated here but will be mentioned in future research.

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